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When Worlds Collide: Different Comparative Static Predictions of Continuous and Discrete Agent Models with Land*

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Abstract

This paper presents a difference in the comparative statics of general equilibrium models with land when there are finitely many agents, and when there is a continuum of agents. Restricting attention to quasi-linear and Cobb-Douglas utility, it is shown that with finitely many agents, an increase in the (marginal) commuting cost increases land rent per unit (that is, land rent averaged over the consumer's equilibrium parcel) paid by the consumer located at each fixed distance from the central business district. In contrast, with a continuum of agents, average land rent goes up for consumers at each fixed distance close to the central business district, is constant at some intermediate distance, and decreases for locations farther away. Therefore, there is a qualitative difference between the two types of models, and this difference is potentially testable.

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1 Introduction

Models with a continuum of consumers are often employed for reasons of mathematical convenience or simplicity. Moreover, they can make precise the notion of perfect competition. As the number of agents in the world is finite, models with an infinite number of agents are not realistic unless they are close to models with a finite number of agents, in terms of equilibria and comparative statics. The scattered literature on general equilibrium models with land has tried to investigate the similarity or dissimilarity between the equilibria of these two types of models. This line of inquiry has met with limited success only; see McLean and Muench (1981), Berliant (1985, 1991), Asami et al. (1991), Kamecke (1993), Papageorgiou and Pines (1990), and Berliant and ten Raa (1991). The intuition for the dissimilarity between the models is that any partition of a σ-finite measure space, such as a Euclidean space, can have only countably many elements of positive measure. So except for a negligible set of consumers out of a continuum, all must consume or even be endowed with a set of measure zero. A corollary is that economies with a finite number of consumers approximating these continuum economies must have land consumption or endowments tending to zero almost surely.

What is the economic significance of this issue? To address this question, it is important to distinguish between models and the real world. Regarding models, the ones with land and a continuum of consumers have an inconsistency embedded in them; see Berliant (1985). Regarding the real world, when we test a model’s implications with real data, we are necessarily testing implications for its analog with a finite number of consumers. Thus, if a model with a continuum of agents differs from its analog with a finite number of agents, either in terms of equilibria or comparative statics, then empirical analysis of a model with a continuum of agents is necessarily suspect, because data for a world with a continuum of agents is unavailable.

This paper presents a dissimilarity in the comparative statics of these two types of models in the cases of quasi-linear and Cobb-Douglas utility functions. The comparative static of interest here is the effect of a change in marginal commuting cost on the per unit land rent paid, averaged over a consumer’s equilibrium parcel, for the consumer who owns land at a given distance from the Central Business District (CBD). The model considered here is the standard closed city model (with an exogenous CBD and an endogenous city boundary). It is shown that when the number of agents is finite, an increase in the (marginal) commuting cost increases average land rent paid by the consumer owning land at any given distance from the CBD. In contrast, the canonical result

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when there is a continuum of agents is that average land rent goes up for consumers who own land at locations close to the CBD, is constant at some intermediate distance, and decreases for consumers farther away; see, for example, Fujita (1989, p. 81, Proposition 3.14, part (iii)).

This is important for both urban economic theory and empirical work. On the theoretical front, this result shows that models with a finite number of consumers are qualitatively different from models with a continuum of consumers, and therefore, in general, it is impossible to conclude that their equilibria are similar. On the empirical front, this result provides a potentially testable prediction. We shall discuss the empirical implications in the conclusions below.

Recent literature on city formation, for example Lucas and Rossi-Hansberg (2002), or the new economic geography, for example Fujita and Thisse (2002), generally employ a continuum of consumers and ordinarily have land as a commodity at least implicitly. We have postulated in our work an exogenously given CBD. In most models of city formation, the CBD or location of firms is endogenous, and there is an agglomeration externality used to determine these locations. However, these models all have embedded in them a model of consumer location and commuting, making our analysis relevant. For example, conditional on the spatial distribution of firms, one might want to consider the consumer location problem.

It is important to be precise about the particular comparative static we consider. This comparative static is exactly the same as the standard one considered by Fujita for the continuous agent model. Specifically, in either model consider a given distance from the CBD such that land at that distance is not used for agriculture. The person or persons residing at that distance from the CBD in equilibrium pays a certain average price for land (where the average is taken over their entire parcel). Now change the marginal commuting cost. If land at that fixed distance from the CBD is not used for agriculture, examine the average price for land paid by the consumer who lives there in equilibrium. This consumer might not be the same as the person who lived there in the equilibrium under the old commuting cost. How does the average price of land change? *We claim that this is the comparative static of empirical relevance, since land price as a function of distance from the CBD is generally what is observed.* Moreover, since we do not observe land prices interior to the parcel, but rather the price function averaged over the entire parcel, we study the latter. In other words, the genesis of this project was to construct the discrete model analog of Fujita's (1989) comparative static, rather than begin with our comparative static and construct the analog in the continuous agent model.

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As pointed out by several readers, there is another, closely related comparative static that has the same sign for both the continuous and discrete agent models. Consider the following exercise. In either the continuous or discrete model, fix a commuting cost and equilibrium. Number the consumers from the CBD outward;[1] each pays a certain average price for their land parcel. Now change the marginal commuting cost. There is a new equilibrium. Since all consumers are identical, they can be numbered arbitrarily from the CBD outward. Maintain the order from the old equilibrium. In general, for the linear city model, the change in the average price paid in equilibrium by each consumer in the continuum and the finite models has the same sign as the change in the marginal commuting cost. We have presented this alternative comparative static for clarity, to emphasize what we are doing in this paper and what we are not doing. It is our view that this second comparative static is not of empirical relevance, since it is difficult to keep track of agents’ locations after a change in commuting cost. In contrast, it is easy to keep track of the per unit price of a parcel at a given location, though the inhabitants may change with a change in commuting cost. It is clearly the case that there are comparative statics that will agree for the two models, in sign if not in magnitude, and others that differ.

The intuition behind our result is as follows. Fix an arbitrary inhabited location in the city, but close to the city edge. For the model with a continuum of agents, when commuting cost rises, in equilibrium the density of agents at the given location goes down, as they relocate to locations closer to the CBD. Thus, demand for the fixed supply of land at that location is reduced, so price goes down. In the model with a finite number of consumers, in equilibrium the number of consumers demanding land at that location is always 1. There is no margin on which to adjust density. The consumer who owns land at that location is squeezed by the increased commuting cost. If the income effect on consumption good is small, expenditure on land must go down, or its average price goes up whereas parcel size shrinks.

As a referee has pointed out, we demonstrate a difference in a comparative static between the model with a continuum of consumers and the model with a finite number of consumers. As enumerated above, many have tried various different methods to connect the two models. Some attempts involve trying to show directly that models with a continuum of agents and models with a fixed, finite number of agents have the same equilibria. Our results address these attempts.

[1] There are some technical issues involved in such a numbering for the continuous model, but they will be ignored here.

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The more classical literature tries to connect the two types of models by approximating the model with a continuum of agents by taking the limit of models with a finite number of agents, where the number of agents tends to infinity. Our results also apply here, since they are independent of the number of agents in the finite model.

In the next section we introduce the notation and present the comparative static in the case of quasi-linear and Cobb-Douglas utility for the model with a finite number of consumers. This is essentially the model of Berliant and Fujita (1992) but with an endogenous city boundary that is determined using an exogenous agricultural land rent. The last section presents our conclusions. An appendix contains a complementary theorem on existence of equilibrium for the closed city model where the extent of the city is endogenous and determined by agricultural land rent.[2]

2 Increasing Rents per Unit Parcel

2.1 Notation

Consider the standard general equilibrium model with a linear city (the CBD at 0) and endogenous city size, l. Suppose there are $N \geq 2$ agents. Each agent's utility function is the same and is given by $u(s, z)$, where s is the size of their land parcel and z is consumption of a composite numéraire. Land is assumed to be a normal good. In addition to (s, z), an agent chooses the location of their lot, where the front door is at distance x from the CBD. Each agent has the same endowment w of the numéraire. In order to consume z, an agent has to commute to the CBD to earn income. The exogenous cost of commuting is $t > 0$ per unit distance to the CBD, and is measured in terms of the numéraire. For the discrete model that is the focus of this paper, the distance to the CBD is measured from the front door location, x. Land price per unit is given by a density p. Agricultural rent outside the city is given by $\xi > 0$.

As usual in this model, an agent's budget constraint is given by $z + \int_x^{x+s} p(s)ds \leq w - tx$. In other words, total spending of an agent on consumption of z units of the numéraire, and a lot of size s at distance x from CBD, is less than or equal to the agent's endowment less the commuting cost tx.

[2] Although equilibrium in the quasi-linear utility case is found explicitly, the Cobb-Douglas case yields a model with no known result on existence of equilibrium. Of course, without such a result the comparative static could be vacuous.

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An equilibrium is given by a collection $(s_n^*, z_n^*)_{n=1}^N$, and a price density function p such that consumers are optimizing and markets clear. Agent marginal rates of substitution, land price, and agricultural rent determine the city size endogenously, as the sum of individual parcel sizes.

2.2 Equilibrium Parcels and Their Comparative Statics

As in Berliant and Fujita (1992), let $Z(s, u)$ be the level of consumption required to achieve utility u, when lot size is s. That is, $Z(\cdot, u)$ is the equation of the indifference curve for utility u. Let $\zeta(\cdot, \cdot) \equiv -Z(\cdot, \cdot)$. Then, using results from Berliant and Fujita (1992), $\zeta_s > 0$, $\zeta_{ss} < 0$, $\zeta_u < 0$, and $\zeta_{su} > 0$. Notice that $\zeta_s(s, u)$ is the (negative of) slope of an indifference curve, and hence it is a marginal rate of substitution.

Consider an equilibrium allocation $(s_n^*, z_n^*)_{n=1}^N$, and let equilibrium utility levels be $(u_n^*)_{n=1}^N$. Notice that as agents have the same endowments and utility function, equilibrium utility levels are identical: $u_1^* = u_2^* = \cdots = u_N^*$. As usual, we label agents by their distance from the CBD, with agent 1 being closest to the CBD and agent N being farthest away from the CBD. Moreover, as shown in Berliant and Fujita (1992), an equilibrium land price function is one that is monotonically non-increasing over distance from the CBD, and has the following form. Over agent 1's parcel, the price is constant at that person's equilibrium MRS; over agent 2's parcel, at the front end, the price decreases as the MRS of agent 1 until it hits the level of agent 2's MRS, and then stays at the level of agent 2's equilibrium MRS; over agent 3's parcel, at the front end, the price decreases as the MRS of agent 2 until it hits the level of agent 3's equilibrium MRS, and then stays at the level of agent 3's equilibrium MRS; and so on.

As usual, first order conditions imply that in equilibrium,

$$\begin{aligned} \zeta_s(s_n^*, u_n^*) &= \zeta_s(s_{n+1}^*, u_{n+1}^*) + t, && \text{for } n = 1, \ldots, N-1, \quad \text{and} \\ \zeta_s(s_n^*, u_n^*) &= \xi && \text{for } n = N. \end{aligned}$$

In particular, $\zeta_s(s_N^*, u_N^*) = \xi$ implies that $\zeta_s(s_{N-1}^*, u_{N-1}^*) = \xi + t$, and proceeding inductively, $\zeta_s(s_{N-k}^*, u_{N-k}^*) = \xi + kt$, for $k = 0, \ldots, N-1$. Changing index yields

$$\zeta_s(s_n^*, u_n^*) = \xi + (N-n)t \quad n = 1, \ldots, N. \tag{1}$$

This provides a relationship between equilibrium marginal rates of substitution in terms of the exogenous parameter of interest t. The relationship is helpful in proving the main result in this

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paper. Toward that goal, the comparative statics of the equilibrium parcel sizes are computed first, as follows.

The relationship (1) implies that for $n = 1, \ldots, N$,

$$N - n = \frac{\partial}{\partial t}\zeta_s(s_n^*, u_n^*) = \zeta_{ss}(s_n^*, u_n^*)\frac{\partial s_n^*}{\partial t} + \zeta_{su}(s_n^*, u_n^*)\frac{\partial u_n^*}{\partial t}.$$

Equation (1) helps determine how equilibrium parcel size changes with respect to t, as follows. As $\zeta_{ss} < 0$, and $\zeta_{su} > 0$, it follows that for $n = 1, \ldots, N$,

$$\frac{\partial u_n^*}{\partial t} < 0 \quad \Rightarrow \quad \frac{\partial s_n^*}{\partial t} < 0.$$

Recall that $u_1^* = u_2^* = \cdots = u_N^*$. Moreover, as land is a normal good, for agent 1 (closest to the CBD), $\frac{\partial u_1^*}{\partial t} < 0$. These observations imply that for $n = 1, \ldots, N$,

$$\frac{(N-n) - \zeta_{su}(s_n^*, u_n^*)\frac{\partial u_n^*}{\partial t}}{\zeta_{ss}(s_n^*, u_n^*)} \quad = \quad \frac{\partial s_n^*}{\partial t} \quad < \quad 0.$$

Thus, for each agent, as commuting cost increases, the equilibrium parcel size, (and therefore, city size) decreases.

2.3 Equilibrium Prices and Rents

For notational convenience, write $u_1^* = u_2^* = \cdots = u_N^* \equiv u$, and write s_n^* as s_n. With this convention, as is well-known, the equilibrium price density is as follows.

$$p(s) = \begin{cases} \zeta_s(s_1, u) & \text{on } [0, s_1] \\ \zeta_s(s - \sum_{k=1}^{n-1} s_k, u) & \text{on } \left[\sum_{k=1}^{n} s_k, \sum_{k=1}^{n-1} s_k + s_{n+1}\right] \qquad n = 1, \ldots, N-1, \\ \zeta_s(s_{n+1}, u) & \text{on } \left[\sum_{k=1}^{n-1} s_k + s_{n+1}, \sum_{k=1}^{n+1} s_k\right] \qquad n = 1, \ldots, N-1, \\ & \text{where, for } n = 1,\ \sum_{k=1}^{n-1} s_k \equiv 0. \end{cases}$$

Define the total land rent paid by consumer n to be

$$\text{rent}_n = \int_{\sum_{k=1}^{n-1} s_k}^{\sum_{k=1}^{n} s_k} p(s)ds$$

2.4 Comparative Statics of Rent per Unit Parcel

This subsection presents the main comparative static result. Fix a distance x from the CBD. Suppose, with only a slight loss of generality, that there is some consumer n such that at the initial equilibrium x is interior to the parcel. We inquire how average land rent, or rent per unit, for the consumer that owns land at x changes with respect to transport cost; that is, $\frac{\partial}{\partial t}\left(\frac{\text{rent}_n}{s_n}\right)$.

Theorem 1: *If utility is quasi-linear, $u(s,z) = v(s)+z$ (where v is increasing and concave), or if utility is Cobb-Douglas, $u(s,z) = s^\alpha z^{1-\alpha}$ (where $\alpha \in (0,1)$), then for $n = 1,...,N$, $\frac{\partial}{\partial t}(\frac{\text{rent}_n}{s_n}) > 0$. Hence, almost surely for $x \in [0,l)$ (namely for x not equal to the front or back of any initial equilibrium parcel) the average price paid by the consumer who owns land at x rises.*[3]

Proof: Notice that $\text{rent}_1 = \zeta_s(s_1,u)s_1$, and therefore,

$$\frac{\partial}{\partial t}\left(\frac{\text{rent}_1}{s_1}\right) \;=\; \frac{\partial}{\partial t}\zeta_s(s_1,u) \;=\; \frac{\partial}{\partial t}(\xi + (N-1)t) \;=\; N-1 \;>\; 0.$$

Therefore, for the first agent, rent per unit increases with commuting cost. Moreover, for $n = 1,\ldots,N-1$,

$$\begin{aligned}
\text{rent}_{n+1} &= \int_{\sum_{k=1}^{n} s_k}^{\sum_{k=1}^{n-1} s_k + s_{n+1}} \zeta_s(s - \textstyle\sum_{k=1}^{n-1} s_k, u)ds + \int_{\sum_{k=1}^{n-1} s_k + s_{n+1}}^{\sum_{k=1}^{n+1} s_k} \zeta_s(s_{n+1},u)ds \\
&= \zeta(s_{n+1},u) - \zeta(s_n,u) + \zeta_s(s_{n+1},u)s_n \\
&= \zeta(s_{n+1},u) - \zeta(s_n,u) + (\xi + (N-n-1)t)s_n.
\end{aligned}$$

Consequently, for $n = 1,\ldots,N-1$,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\text{rent}_{n+1}}{s_{n+1}}\right) \\
&= \frac{1}{s_{n+1}^2} s_{n+1}\left[\zeta_s(s_{n+1},u)\frac{\partial s_{n+1}}{\partial t} + \zeta_u(s_{n+1},u)\frac{\partial u}{\partial t} - \zeta_s(s_n,u)\frac{\partial s_n}{\partial t} - \zeta_u(s_n,u)\frac{\partial u}{\partial t}\right] \\
&\quad - \frac{1}{s_{n+1}^2}\left[\zeta(s_{n+1},u) - \zeta(s_n,u)\right]\frac{\partial s_{n+1}}{\partial t} \\
&\quad + \frac{1}{s_{n+1}^2} s_{n+1}\left[s_n(N-n-1) + \zeta_s(s_{n+1},u)\frac{\partial s_n}{\partial t}\right] \\
&\quad - \frac{1}{s_{n+1}^2}\zeta_s(s_{n+1},u)s_n\frac{\partial s_{n+1}}{\partial t} \\
&= \frac{1}{s_{n+1}^2} s_{n+1}\left[\zeta_u(s_{n+1},u) - \zeta_u(s_n,u)\right]\frac{\partial u}{\partial t} \\
&\quad + \frac{1}{s_{n+1}^2} s_{n+1}s_n(N-n-1) \\
&\quad + \frac{1}{s_{n+1}^2} s_{n+1}\left[\zeta_s(s_{n+1},u) - \zeta_s(s_n,u)\right]\frac{\partial s_n}{\partial t} \\
&\quad + \frac{1}{s_{n+1}^2}\left[\zeta_s(s_{n+1},u)(s_{n+1}-s_n) - (\zeta(s_{n+1},u) - \zeta(s_n,u))\right]\frac{\partial s_{n+1}}{\partial t}.
\end{aligned}$$

[3] Technically, if the land parcel of consumer n is represented as $[x_n, x_n + s_n)$, then the comparative static for an increase in the marginal commuting cost works fine except at l. However, for a decrease in the marginal commuting cost, we must exclude the boundary points between parcels.

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As lot sizes are positive, the above relationship implies that for $n = 1, \ldots, N-1$,

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\text{rent}_{n+1}}{s_{n+1}}\right) > 0 \Leftrightarrow \quad & s_{n+1}s_n(N-n-1) \; + \; s_{n+1}\left[\zeta_u(s_{n+1},u) - \zeta_u(s_n,u)\right]\frac{\partial u}{\partial t} \\ & + s_{n+1}\left[\zeta_s(s_{n+1},u) - \zeta_s(s_n,u)\right]\frac{\partial s_n}{\partial t} \\ & + \left[\zeta_s(s_{n+1},u)(s_{n+1}-s_n) - (\zeta(s_{n+1},u) - \zeta(s_n,u))\right]\frac{\partial s_{n+1}}{\partial t} \quad > \quad 0. \end{aligned} \tag{2}$$

Notice that the first term on the right-hand side is non-negative, the third term is positive because $\zeta_{ss} < 0$ implies that $\zeta_s(s_{n+1},u) - \zeta_s(s_n,u) < 0$ and that $\frac{\partial s_n}{\partial t} < 0$, and the fourth term is positive because $\zeta(\cdot,u)$ is concave and $\frac{\partial s_{n+1}}{\partial t} < 0$. In general, the second term is non-positive, because $\zeta_{su} \geq 0$ implies that ζ_u is (weakly) increasing in s, $s_n < s_{n+1}$, and $\frac{\partial u}{\partial t} < 0$. Thus, in general, it is possible that the expression on the right-hand size is not positive. However, as documented next, the expression on the right-hand size is positive for the frequently-used classes of quasi-linear and Cobb-Douglas utility.

For quasi-linear utility, the second term above equals zero, as follows. Write $u(s,z) = v(s) + z$, where v is increasing and concave, and notice that $\zeta(s,u) = v(s) - u$. Consequently, $\zeta_u = -1$ and $\zeta_{su} = 0$. In particular, ζ_u does not depend on s, and the second term above equals zero. Therefore, in the case of quasi-linear utility, for every n, $\frac{\partial}{\partial t}\left(\frac{\text{rent}_n}{s_n}\right) > 0$.

For Cobb-Douglas utility, the entire expression above is positive, as follows. For notational convenience, the argument u in the functions ζ, ζ_s, ζ_{ss}, ζ_u, and ζ_{su} is suppressed for now. Notice that for $n = 1, \ldots, N-1$, $s_{n+1}s_n(N-n-1) \geq 0$ implies that in order to conclude that for $n = 1, \ldots, N-1$, $\frac{\partial}{\partial t}\left(\frac{\text{rent}_{n+1}}{s_{n+1}}\right) > 0$, it is sufficient to show that

$$\begin{aligned} & \left[\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))\right]\frac{\partial s_{n+1}}{\partial t} \\ & + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right]\frac{\partial s_n}{\partial t} \\ & + \quad s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right]\frac{\partial u}{\partial t} \quad > \quad 0. \end{aligned}$$

Using the equilibrium relationship that for $n = 1, \ldots, N$,

$$\frac{\partial s_n}{\partial t} = \frac{(N-n) - \zeta_{su}(s_n)\frac{\partial u}{\partial t}}{\zeta_{ss}(s_n)},$$

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the expression on the left-hand side of the above inequality can be written as follows.

$$\begin{aligned}
&[\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{(N-n-1)-\zeta_{su}(s_{n+1})\frac{\partial u}{\partial t}}{\zeta_{ss}(s_{n+1})} \\
&\quad + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{(N-n)-\zeta_{su}(s_n)\frac{\partial u}{\partial t}}{\zeta_{ss}(s_n)} \\
&\quad + \quad s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right] \tfrac{\partial u}{\partial t} \\
&= [\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{(N-n-1)}{\zeta_{ss}(s_{n+1})} \\
&\quad + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{(N-n)}{\zeta_{ss}(s_n)} \\
&\quad + \quad [\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{-\zeta_{su}(s_{n+1})}{\zeta_{ss}(s_{n+1})} \tfrac{\partial u}{\partial t} \\
&\quad + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{-\zeta_{su}(s_n)}{\zeta_{ss}(s_n)} \tfrac{\partial u}{\partial t} \\
&\quad + \quad s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right] \tfrac{\partial u}{\partial t}.
\end{aligned}$$

For this last expression, notice that the first term is non-negative, and the second term is positive. In the case of Cobb-Douglas utility, the sum of the remaining three terms equals zero, as shown below.

Consider the utility function, $u(s,z) = s^{\alpha} z^{1-\alpha}$. Then

$$\begin{array}{lll}
\zeta(s,u) = -\frac{u^{\frac{1}{1-\alpha}}}{s^{\frac{\alpha}{1-\alpha}}} & \zeta_s(s,u) = \frac{\alpha}{1-\alpha}\frac{u^{\frac{1}{1-\alpha}}}{s^{\frac{1}{1-\alpha}}} & \zeta_{ss}(s,u) = -\frac{\alpha}{(1-\alpha)^2}\frac{u^{\frac{1}{1-\alpha}}}{s^{\frac{1}{1-\alpha}+1}} \\
\zeta_u(s,u) = -\frac{1}{1-\alpha}\frac{u^{\frac{\alpha}{1-\alpha}}}{s^{\frac{\alpha}{1-\alpha}}} & \zeta_{su}(s,u) = \frac{\alpha}{(1-\alpha)^2}\frac{u^{\frac{\alpha}{1-\alpha}}}{s^{\frac{1}{1-\alpha}}} & \frac{\zeta_{su}}{\zeta_{ss}}(s,u) = -\frac{s}{u}.
\end{array}$$

Therefore,

$$\begin{aligned}
& s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right] \quad - \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{\zeta_{su}(s_n)}{\zeta_{ss}(s_n)} \\
&\quad - \quad [\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{\zeta_{su}(s_{n+1})}{\zeta_{ss}(s_{n+1})} \\
&= -\tfrac{1}{1-\alpha} u^{\frac{\alpha}{1-\alpha}} s_{n+1} \left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) + \tfrac{\alpha}{1-\alpha} u^{\frac{1}{1-\alpha}} \left(\frac{1}{s_{n+1}^{\frac{1}{1-\alpha}}} - \frac{1}{s_n^{\frac{1}{1-\alpha}}} \right) \frac{s_{n+1}s_n}{u} \\
&\quad + \left[\tfrac{\alpha}{1-\alpha}(s_{n+1}-s_n) \frac{u^{\frac{1}{1-\alpha}}}{s_{n+1}^{\frac{1}{1-\alpha}}} + u^{\frac{1}{1-\alpha}} \left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) \right] \frac{s_{n+1}}{u} \\
&= -\frac{u^{\frac{\alpha}{1-\alpha}}}{1-\alpha} s_{n+1} \left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) + \tfrac{\alpha}{1-\alpha} u^{\frac{\alpha}{1-\alpha}} \frac{s_n s_{n+1}}{s_{n+1}^{\frac{1}{1-\alpha}}} - \tfrac{\alpha}{1-\alpha} u^{\frac{\alpha}{1-\alpha}} \frac{s_{n+1}}{s_n^{\frac{\alpha}{1-\alpha}}} \\
&\quad + \tfrac{\alpha}{1-\alpha} u^{\frac{\alpha}{1-\alpha}} \frac{s_{n+1}}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \tfrac{\alpha}{1-\alpha} u^{\frac{\alpha}{1-\alpha}} \frac{s_n s_{n+1}}{s_{n+1}^{\frac{1}{1-\alpha}}} + u^{\frac{\alpha}{1-\alpha}} s_{n+1} \left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) \\
&= -\frac{u^{\frac{\alpha}{1-\alpha}}}{1-\alpha} s_{n+1} \left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) + u^{\frac{\alpha}{1-\alpha}} s_{n+1} \left(-\tfrac{\alpha}{1-\alpha} \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} + \tfrac{\alpha}{1-\alpha} \frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} + \frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) \\
&= u^{\frac{\alpha}{1-\alpha}} \left[-\frac{s_{n+1}}{1-\alpha} \left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) + \frac{s_{n+1}}{1-\alpha} \left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} \right) \right] \\
&= 0.
\end{aligned}$$

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Consequently,

$$\begin{aligned}&[\zeta_s(s_{n+1})(s_{n+1}-s_n)-(\zeta(s_{n+1})-\zeta(s_n))]\,\tfrac{-\zeta_{su}(s_{n+1})}{\zeta_{ss}(s_{n+1})}\tfrac{\partial u}{\partial t}\\ &\quad+\quad s_{n+1}\,[\zeta_s(s_{n+1})-\zeta_s(s_n)]\,\tfrac{-\zeta_{su}(s_n)}{\zeta_{ss}(s_n)}\tfrac{\partial u}{\partial t}\\ &\quad+\quad s_{n+1}\,[\zeta_u(s_{n+1})-\zeta_u(s_n)]\,\tfrac{\partial u}{\partial t}\\ &=0.\end{aligned}$$

Therefore, in the case of Cobb-Douglas utility, for every n, $\frac{\partial}{\partial t}\left(\frac{\mathrm{rent}_n}{s_n}\right) > 0$. For any x in the interior of a parcel, the person who purchases land at location x in equilibrium does not change with a small change in t.■

It is useful at this point to make some remarks about the proof and possible extensions. The important equation to focus on to make the results more general is (2); in particular, the second term on the right hand side is possibly negative, and in this case the result might not hold. In particular, a sufficient condition for the second term to be small in absolute value is that $\mid \zeta_u(s,u)\frac{\partial u}{\partial t} \mid$ is small, or that the income effect for numéraire is small. A referee has noted that since the income effect for quasi-linear utility is zero and that the income and substitution effects for Cobb-Douglas utility cancel, that is the reason the sum of terms beyond the first must be zero. Although, given the values of other exogenous parameters, presuming a small enough income effect is sufficient, other variables do matter. For example, the second term involves a factor s_{n+1}, and thus agricultural land rent plays an indirect role in the sign of (2). Moreover, there is more slack in the equation due to the first term, since it is always positive. It seems that the necessary and sufficient conditions for our result to hold are likely to be complicated. Alternatively, replacing functional forms with a small income effect in the sufficiency assumptions of our theorem is likely to exclude Cobb-Douglas utility.

3 Conclusions

We have examined a comparative static in closed city models with an endogenous city boundary and a finite number of consumers. We have compared it to the analogous comparative static derived in Fujita (1989) for the model with a continuum of consumers, and we have found a difference. For researchers in urban economic theory, the implication is that there are qualitative differences between the models. For empiricists, the possibility of testing the models against one another is real.

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Although the quasi-linear and Cobb-Douglas utility cases are sufficient to make our point that the comparative statics in the model with a finite number of consumers and the model with a continuum of consumers can differ,[4] the analogous comparative static for general utility functions seems difficult, or at least algebraically burdensome. But even the result for Cobb-Douglas utility must be backed up by a theorem on existence of equilibrium for the finite model with an endogenous city boundary.[5] We provide this theorem in a brief appendix below.

Which model, finite or continuum of agents, will be verified empirically? Probably this depends on the context. One obvious way to test the models is to look at the per unit cost of land parcels at each fixed distance from the CBD in a city, say Chicago, before and after a change in commuting cost, say the introduction of a new "el" line. The work of McMillen and McDonald (2004) should be useful. Our model does not account for firm relocation and its impact on the comparative static, so this must be controlled for in empirical applications.

It is unclear if the difference in the comparative static presented extends to other comparative statics as well. However, we conjecture that comparative statics that exploit the general notion that the equilibrium density of agents at a fixed location can change in the model with a continuum of agents but not for the finite model can differ in sign or magnitude.

4 Appendix

Theorem 1 *Under standard regularity assumptions on the utility function (Berliant and Fujita, 1992, Assumption 1) there exists an equilibrium.*

Proof: For $p_1 \equiv \xi + (N-1)t$, construct $x_{n+1}(p_1)$, the sum of consumers' Marshallian demand for land when consumer 1 (the consumer closest to the CBD) faces price p_1, as in Berliant and Fujita (1992, pp. 561-562). Set $l = x_{n+1}(p_1)$. Apply Berliant and Fujita (1992, Proposition 4): under Assumption 1 of their paper, for any fixed $l > 0$, there exists an equilibrium. This equilibrium will have the property that the marginal willingness to pay for land of consumer N, the outermost consumer, is equal to the agricultural land rent.

[4] It is easy to verify that the result in Fujita (1989) applies to quasi-linear utility.

[5] Notice that existence of equilibrium in the case of quasi-linear utility is not an issue, since we can find the equilibrium explicitly using first order conditions, and thus we have proved that it exists. For more detail and a graphical depiction of the finite model, see Berliant and LaFountain (2006).

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Open Access version: http://kuscholarworks.ku.edu/dspace/.

References

[1] Asami, Y., Fujita, M., Smith, T., 1991. On the foundations of land use theory: discrete versus continuous populations. Regional Science and Urban Economics 20, 473-508.

[2] Berliant, M., 1985. Equilibrium models with land: a criticism and an alternative. Regional Science and Urban Economics 15, 325-340.

[3] Berliant, M., 1991. Comments on "On the foundations of land use theory: discrete versus continuous populations" by Y. Asami, M. Fujita and T. Smith. Regional Science and Urban Economics 21, 639-45.

[4] Berliant, M., Fujita, M., 1992. Alonso's discrete population model of land use: efficient allocations and competitive equilibria. International Economic Review 33, 535-566.

[5] Berliant, M., LaFountain, C., 2006. Space in general equilibrium. In: Arnott, R., McMillen, D. (Eds.). A Companion to Urban Economics. Williston, VT: Blackwell, 109-127.

[6] Berliant, M., ten Raa, T., 1991. On the continuum approach of spatial and some local public goods or product differentiation models: some problems. Journal of Economic Theory 55, 95-120.

[7] Fujita, M., 1989. Urban Economic Theory. Cambridge: Cambridge University Press.

[8] Fujita, M., Thisse, J.-F., 2002. Economics of Agglomeration. Cambridge: Cambridge University Press.

[9] Kamecke, U., 1993. Mean city: a consistent approximation of bid rent equilibria. Journal of Urban Economics 33, 48-67.

[10] Lucas, R.E., Jr., Rossi-Hansberg, E., 2002. On the internal structure of cities. Econometrica 70, 1445-1476.

[11] McLean, R., Muench, T., 1981. Approximate decentralization of the Beckmann-Koopmans assignment/shipping problem for a large discrete Mills city and its limit relationship to the continuum Mills city. Cowles Foundation Working Paper No. 236.

[12] McMillen, D.P., McDonald, J.F., 2004. Reaction of house prices to a new rapid transit line: Chicago's midway line, 1983-1999. Real Estate Economics 32, 463-486.

[13] Papageorgiou, Y.Y., Pines, D., 1990. The logical foundations of urban economics are consistent. Journal of Economic Theory 50, 37-53.